MULTI-GRADED EXTENDED REES ALGEBRAS OF $\mathfrak{m}$-PRIMARY IDEALS

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1. Introduction

In this paper we consider multi-graded extended Rees algebras of zero dimensional ideals which are Cohen-Macaulay (CM) with minimal multiplicity. We show that the minimal multiplicity property can occur only for the ordinary extended Rees algebra and the bigraded extended Rees algebra. For the bigraded extended Rees algebra we find necessary conditions for it to be CM with minimal multiplicity. We also produce bigraded Rees algebras which are Cohen-Macaulay with minimal multiplicity.

A considerable amount was known for the ordinary extended Rees algebra. Among the many we quote ([KV], [Ve1], [Ve2], [Ve4]). There was nothing known concerning the minimal multiplicity of the multi-graded extended Rees algebra. One of the crucial results needed was the formula of multiplicity of a maximal homogeneous ideal. This formula was obtained in the author in [D2].

Throughout this paper $(R, \mathfrak{m})$ will denote a Noetherian local ring of positive dimension. Without loss of generality we will assume that $R/\mathfrak{m}$ is infinite. It is well-known that for any CM local ring $(R, \mathfrak{m})$, $e(\mathfrak{m}) \geq \mu(\mathfrak{m}) - \dim\ R + 1$, where $e(\mathfrak{m})$ denotes the multiplicity of $\mathfrak{m}$ and $\mu(\mathfrak{m})$ is the minimal number of generators of $\mathfrak{m}$. A CM local ring is said to have *minimal multiplicity* if equality holds.

Let $I_1, \ldots, I_g$ be ideals of positive height in $(R, \mathfrak{m})$ and let $t_1, \ldots, t_g$ be indeterminates. The *multi-graded extended Rees algebra* of R with respect to the ideals $I_1, \ldots, I_g$ is the graded ring $\mathcal{B}(\mathbf{I}) := \oplus_{r_j \in \mathbb{Z}, 1 \leq j \leq g} (I_1 t_1)^{r_1} \cdots (I_g t_g)^{r_g}$. Here $I_j^{r_j} = R$, if $r_j \leq 0$ for all $j = 1, \ldots, g$. Let $\mathcal{N}$ denote the maximal homogeneous ideal of $\mathcal{B}(\mathbf{I})$. The *multi-Rees algebra* is the graded ring $\bigoplus_{r_i \geq 0} (I_1 t_1)^{r_1} \cdots (I_g t_g)^{r_g}$ and will be denoted by $\mathcal{R}(\mathbf{I})$.

In the past decade several researchers have investigated the multi-Rees algebra. Since the multi-Rees algebra is a subring of the multi-graded extended Rees algebra, it is natural

to expect them to have similar ring-theoretic properties. However, there was no progress concerning the multi-graded extended Rees algebra.

Hence we will briefly state some of the earlier known results on the Rees algebra and the extended Rees algebra. It is well-known that if I is an ideal of positive height in a CM local ring R and if $\mathcal{R}(I)$ is CM, then the *associated graded ring* $G(I) := \oplus_{r\geq 0} I^r/I^{r+1}$ is also CM [Hu, Proposition 1.1]. It is easy to see that $G(I)$ is CM if and only if the extended Rees ring $\mathcal{B}(I)$ is. In 1989, Verma showed that if R is a CM local ring of dimension two with minimal multiplicity, then for all positive integers r, $\mathcal{R}(\mathfrak{m}^r)$ and $\mathcal{B}(\mathfrak{m}^r)$ are CM with minimal multiplicity [Ve1, Theorem 3.3, 4.3]. In the same year he showed that if I is a parameter ideal in a CM ring of dimension at least two and if $\ell(I+\mathfrak{m}^2/\mathfrak{m}^2) \geq \dim\ R-1$, then $\mathcal{R}(I)$ and $\mathcal{B}(I)$ are CM with minimal multiplicity [Ve2, Theorem 3.1, 3.2]. In 1991 he proved the following: Let $(R,\mathfrak{m})$ be a regular local ring of dimension two. Let I be a contracted $\mathfrak{m}$-primary ideal with reduction number one. Then $\mathcal{R}(I)$ and $\mathcal{B}(I)$ are CM with minimal multiplicity [Ve4, Theorem 3.1, 4.3].

In [HHRT] Herrmann et. al. remarked that if I is an ideal of positive height and if $I_1 = \cdots = I_g = I$, then the multi-Rees algebra $\mathcal{R}(\mathbf{I})$ behaves like the ordinary extended Rees algebra $\mathcal{R}(I)$. In this paper they studied the CM property of the multi-Rees algebra. Minimal multiplicity of the multi-Rees algebra has been studied in [Ve5], [HHRT] and [D1].

The following results which were obtained in the authors thesis played an important role in obtaining our results:

(1) A relation between the number of generators of an $\mathfrak{m}$-primary ideal in a CM local ring and a certain mixed multiplicity (Theorem 3.2).
(2) The bounds on the mixed multiplicities of ideals in $(R,\mathfrak{m})$ (see Lemma 3.5, Lemma 3.6).
(3) The bounds on $\ell(I_1+I_2+\mathfrak{m}^2/\mathfrak{m}^2)$ when $\mathcal{B}(\mathbf{I})_{\mathcal{N}}$ is CM with minimal multiplicity, where I_1 and I_2 are $\mathfrak{m}$-primary ideals in a CM ring $(R,\mathfrak{m})$ (see Lemma 4.1, Lemma 4.2).

Remark 1.1. The above mentioned results also give a simple and unified proof for the known results for the ordinary extended Rees algebra. We do not mention these results here. But we answer a question of Verma concerning the ordinary extended Rees algebra (see [Ve1, pg 3015] and Example 5.5). This gives an infinite class of examples of ordinary extended Rees algebras which are Cohen-Macaulay with minimal multiplicity even the original ring does not have minimal multiplicity. It was not possible to easily see or

construct this example with the methods used in Verma's paper concerning the ordinary extended Rees algebra.

We now summarise the main results in this paper. In Section two we prove that for a CM local ring minimal multiplicity can occur only for the ordinary extended Rees algebra and the bigraded extended Rees algebra (i.e. when $g = 1, 2$). In Section three we obtain necessary conditions for the bigraded extended Rees algebra to be CM with minimal multiplicity. In Section four we consider bigraded extended Rees algebras which are CM with minimal multiplicity. We end the paper with an example.

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2. Preliminaries

2.1. An ideal $J \subseteq I$ is a *reduction* of I if there exists a positive integer r such that $JI^r = I^{r+1}$ [NR]. The ideal J is called a *minimal reduction* of I if J is minimal with respect to inclusion among all reductions of I. If $R/\mathfrak{m}$ is infinite, then any minimal reduction of I is generated by $a(I)$ elements, where $a(I) = \dim \bigoplus_{n\geq 0}(I^n/\mathfrak{m}I^n)$ is called *the analytic spread of* I. For an ideal I in R, $ht\ I \leq a(I) \leq \dim\ R$ [R2]. If J is a reduction of I, then the reduction number of I with respect to J is defined to be

$$r_J(I) = \min\{n \geq 0 \mid JI^n = I^{n+1}\}.$$

The *reduction number* of I is defined to be

$$r(I) = \min\{r_J(I) \mid J \text{ is a minimal reduction of } I\}.$$

2.2. Let $I_1, \dots, I_g$ be $\mathfrak{m}$-primary ideals in a local ring R of dimension d. For $r_1, \dots, r_g$ large, $\ell_R(I_1^{r_1} I_2^{r_2} \cdots I_g^{r_g} / I_1^{r_1+1} I_2^{r_2} \cdots I_g^{r_g})$ is a polynomial of degree $d-1$ in $r_1, \dots, r_g$ and can be written in the form

$$\sum_{q_1+\cdots+q_g=d-1} e(I_1^{[q_1+1]}|I_2^{[q_2]}|\cdots|I_g^{[q_g]})\frac{r_1^{q_1}}{q_1!}\cdots\frac{r_g^{q_g}}{q_g!} + \text{lower degree terms;}$$

where $e(I_1^{[q_1+1]}|I_2^{[q_2]}|\cdots|I_g^{[q_g]})$ are positive integers and they are called the *mixed multiplicities* of the set of ideals $\{I_1,\ldots,I_g\}$ [T]. For $g=2$, we will use the notation

$$e_q(I_1|I_2) := e(I_1^{[d-q]}|I_2^{[q]}) \qquad\qquad 0 \leq q \leq d-1.$$

2.3. Rees obtained an interpretation of mixed multiplicities in terms of joint reductions [R3]. Let $I_1,\ldots,I_g$ be $\mathfrak{m}$-primary ideals. A set of elements $x_1,\ldots,x_g$ is a called a *joint reduction* of the set of ideals $\{I_1,\ldots,I_g\}$ if $x_j \in I_j$ for $j=1,\ldots,g$ and if $\sum_{j=1}^{g} x_j I_1\cdots\widehat{I_j}\cdots I_g$ is a reduction of $I_1\cdots I_g$. Rees proved that if $R/\mathfrak{m}$ is infinite, $g=\dim\ R$ and $I_1,\ldots I_g$ are $\mathfrak{m}$-primary ideals, then joint reductions exist [R3]. It follows that if I and J are $\mathfrak{m}$-primary ideals in a local ring $(R,\mathfrak{m})$ then $e_0(I|J)=e(I)$ [R1]. We end this section by stating an important result of Rees.

Lemma 2.4. (Rees' Lemma) [R3, Lemma 1.2] *Suppose $(R,\mathfrak{m})$ is a local ring with infinite residue field. Let $\{I_1,\ldots,I_g\}$ be a set of ideals of R and let $\mathcal{P}$ be a finite collection of prime ideals of R not containing any of $I_1,\ldots,I_g$. Then for each $i=1,\ldots,g$, there exists an element $x_i \in I_i$, x_i not contained in any prime ideal of $\mathcal{P}$ and an integer s_i such that for $r_i \geq s_i$ and for all positive integers $r_1,,\ldots,\widehat{r_i},\ldots,r_g$;*

$$x_iR \cap I_1^{r_1}\cdots I_g^{r_g} = x_i I_1^{r_1}\cdots I_i^{r_i-1}\cdots I_g^{r_g}.$$

3. The case $g \geq 3$

The main result in this section is:

Theorem 3.1. *Let $(R,\mathfrak{m})$ be a CM local ring of dimension d. Let $I_1,\ldots,I_g$ be $\mathfrak{m}$-primary ideals in R. If $\mathcal{B}(\mathbf{I})_{\mathcal{N}}$ is CM with minimal multiplicity, then $g \leq 2$.*

By a result of Valla, $\dim\ \mathcal{B}(\mathbf{I}) = \dim\ R+g$ [Val]. Notice that $e(\mathcal{N}\mathcal{B}(\mathbf{I})_{\mathcal{N}}) = e(\mathcal{N})$ and $\mu(\mathcal{N}\mathcal{B}(\mathbf{I})_{\mathcal{N}}) = \mu(\mathcal{N})$. Hence, if $\mathcal{B}(\mathbf{I})_{\mathcal{N}}$ is CM, then it has minimal multiplicity if and only if $e(\mathcal{N}) = \mu(\mathcal{N}) - (\dim\ R+g)+1$.

We state an interesting inequality which relates the number of generators of an ideal with a certain mixed multiplicity.

Theorem 3.2. *Let $(R,\mathfrak{m})$ be a CM local ring of positive dimension d. Let I be an $\mathfrak{m}$-primary ideal of R. Then*

$$\mu(I) \leq e_{d-1}(\mathfrak{m}|I)+d-1.$$

Proof. We induct on d. The case $d = 1$ has been proved by J. Sally [Sa2, pg 49]. If $d > 1$, then by Lemma 2.4 there exists a non-zero-divisor $y \in I$ and a positive integer s_0 so that for $s \geq s_0$ and $r > 0$,

$$yR \cap \mathfrak{m}^r I^s = y\mathfrak{m}^r I^{s-1}.$$

Let "-" denote the image in $\overline{R} = R/yR$. By induction hypothesis we have

$$\mu(I) \leq \mu(\overline{I}) + 1 \leq e_{d-2}(\overline{\mathfrak{m}} \,|\, \overline{I}) + d - 1 = e_{d-1}(\mathfrak{m}|I) + d - 1.$$

□

An upper bound on the number of generators of the maximal homogeneous ideal of the multi-graded extended Rees algebra can be estimated by Theorem 3.2.

Remark 3.3. Let $I_1, \ldots, I_g$ be $\mathfrak{m}$-primary ideals in a CM local ring $(R, \mathfrak{m})$. Put $L = I_1 + \cdots + I_g + \mathfrak{m}^2$. Comparing the graded components of $\mathcal{N}$ and $\mathcal{N}^2$ we get

$$\begin{aligned} \mu(\mathcal{N}) &= \sum_{j=1}^{g} \ell(R/\mathfrak{m}) + \ell(\mathfrak{m}/L) + \sum_{j=1}^{g} \mu(I_j) \\ &= g + \mu(\mathfrak{m}) + \sum_{j=1}^{g} \mu(I_j) - \ell\left(L/\mathfrak{m}^2\right) \\ &\leq e(\mathfrak{m}) + \sum_{j=1}^{g} e_{d-1}(\mathfrak{m}|I_j) + d(g+1) - 1 - \ell\left(L/\mathfrak{m}^2\right) \text{ [by Theorem 3.2].} \qquad (1) \end{aligned}$$

The multiplicity of $\mathcal{N}$ can be expressed in terms of mixed multiplicities of ideals in R. Hence, any bound on mixed multiplicities of ideals in R will give a bound on the multiplicity of R.

Theorem 3.4. [D2, Theorem 1.2] *Let $I_1, \ldots, I_g$ be $\mathfrak{m}$-primary ideals in $(R, \mathfrak{m})$. Put $L = I_1 + \cdots + I_g + \mathfrak{m}^2$. Then*

$$e(\mathcal{N}) = \frac{1}{2^d}\left[\sum_{n=0}^{g}\sum_{q=0}^{d-1} 2^{d-1-q} \sum_{\substack{q_1+\cdots+q_n=d-1-q \\ 1\leq i_1<\cdots<i_n\leq g}} e(L^{[q+1]}|I_{i_1}^{[q_1]}|\cdots|I_{i_n}^{[q_n]})\right].$$

Lemma 3.5. *Let $(R, \mathfrak{m})$ be a local ring and $I_1, \ldots, I_g$ be $\mathfrak{m}$-primary ideals of R. Then for all nonnegative integers $q_1, \ldots, q_g$ satisfying $q_1 + \cdots + q_g = d - 1$,*

$$e(I_1^{[q_1+1]}|I_2^{[q_2]}|\cdots|I_g^{[q_g]}) \geq e(I_1 + \cdots + I_g).$$

Proof. Since $I_1, \dots, I_g$ are $\mathfrak{m}$-primary, by [R3, Theorem 2.4], there exists a joint reduction $x_1, \dots, x_d$ of q_1+1 copies of I_1, q_2 copies of $I_2, \dots, q_g$ copies of I_g, such that $e(x_1, \dots, x_d) = e(I_1^{[q_1+1]}|I_2^{[q_2]} \dots |I_g^{[q_g]})$. Since $(x_1, \dots, x_d) \subseteq I_1+\cdots+I_g$, $e(x_1, \dots, x_d) \geq e(I_1+\cdots+I_g)$. $\square$

Lemma 3.6. [cf. Sw, Lemma 2.8] *Let $(R, \mathfrak{m})$ be a local ring. Let $I_1, \dots, I_d$ be $\mathfrak{m}$-primary ideals in R. Let $x_i \in I_i$ for $i = 1, \dots, d$ be such that $(x_1, \dots, x_d)$ is $\mathfrak{m}$-primary. Then*

$$e(I_1|\cdots|I_d) \leq e(x_1, \dots, x_d).$$

If $(R, \mathfrak{m})$ is quasi-unmixed and equality holds, then $x_1, \dots, x_d$ is a joint reduction of the set of ideals $\{I_1, \dots, I_d\}$.

Proof of Theorem 3.1: Put $L = I_1 + \cdots + I_g + \mathfrak{m}^2$. Since $\mathcal{B}(\mathbf{I})_{\mathcal{N}}$ is CM with minimal multiplicity, $e(\mathcal{N}) = \mu(\mathcal{N}) - \dim\ \mathcal{B}(\mathbf{I}) + 1$. From Remark 3.3 it follows that

$$\begin{aligned} \mu(\mathcal{N}) - \dim\ \mathcal{B}(\mathbf{I}) + 1 &\leq e(\mathfrak{m}) + \sum_{j=1}^{g} e_{d-1}(\mathfrak{m}|I_j) + g(d-1) - \ell(L/\mathfrak{m}^2) \\ &\leq e(\mathfrak{m}) + \sum_{j=1}^{g} e_{d-1}(\mathfrak{m}|I_j) + g(d-1). \end{aligned} \tag{2}$$

Let $d = 1$. Then

$$e(\mathcal{N}) \leq (g+1)e(\mathfrak{m}) \qquad \text{[from (2)]} \tag{3}$$

$$\text{and} \qquad e(\mathcal{N}) = 2^{g-1}e(L) \geq 2^{g-1}e(\mathfrak{m}) \qquad \text{[from Theorem 3.4].} \tag{4}$$

Clearly, $2^{g-1} > g+1$ for $g > 3$. If $g = 3$, then equality holds in (3) and (4). This implies that $e(L) = e(\mathfrak{m})$ and $L = \mathfrak{m}^2$ which is not possible. Hence $g \leq 2$.

Let $d \geq 2$ and $g \geq 3$. It is enough to show that

$$e(\mathcal{N}) > e(\mathfrak{m}) + \sum_{j=1}^{g} e_{d-1}(\mathfrak{m}|I_j) + g(d-1). \tag{5}$$

Since $I_1, \dots, I_g$ are $\mathfrak{m}$-primary ideals, the mixed multiplicities which appear in the formula of $e(\mathcal{N})$ (see Theorem 3.4) are positive integers. Moreover, $e_{d-1}(L|I_j) \geq e_{d-1}(\mathfrak{m}|I_j)$ for all $j = 1, \dots, g$ (Lemma 3.6). In the multiplicity formula for $e(\mathcal{N})$, if we replace $e_{d-1}(L|I_j)$

by $e_{d-1}(\mathfrak{m}|I_j)$ $(1 \leq j \leq g)$ and the remaining terms by 1 we get

$$\begin{aligned} e(\mathcal{N}) &\geq \frac{1}{2^d}\left[1+\sum_{n=1}^{g}\binom{g}{n}\left[\sum_{q=0}^{d-1}2^q\binom{q+n-1}{n-1}-2^{d-1}n\right]\right]+2^{g-2}\sum_{j=1}^{g}e_{d-1}(\mathfrak{m}|I_j) \\ &= \frac{1}{2^d}\left[1+\sum_{n=1}^{g}\binom{g}{n}\sum_{q=0}^{d-1}2^q\binom{q+n-1}{n-1}\right]-2^{g-2}g+2^{g-2}\sum_{j=1}^{g}e_{d-1}(\mathfrak{m}|I_j). \qquad (6) \end{aligned}$$

Clearly

$$\begin{aligned} & 2^{g-2}\sum_{j=1}^{g}e_{d-1}(\mathfrak{m}|I_j)-2^{g-2}g \\ \geq\ & \sum_{j=1}^{g}e_{d-1}(\mathfrak{m}|I_j)+e(\mathfrak{m})+(2^{g-2}-1)g-1-2^{g-2}g \qquad \text{[by Lemma 3.5]} \\ =\ & \sum_{j=1}^{g}e_{d-1}(\mathfrak{m}|I_j)+e(\mathfrak{m})-g-1. \qquad (7) \end{aligned}$$

We will show by induction on d that

$$1+\sum_{n=1}^{g}\binom{g}{n}\sum_{q=0}^{d-1}2^q\binom{q+n-1}{n-1} > 2^d(gd+1). \qquad (8)$$

If $d=2$, then it is easy to see that the left-hand side of (8) is $2^g(g+1)$ and the right-hand side is $4(2g+1)$. If $d \geq 3$, then

$$\begin{aligned} & 1+\sum_{n=1}^{g}\binom{g}{n}\sum_{q=0}^{d-1}2^q\binom{q+n-1}{n-1} \\ =\ & 1+\sum_{n=1}^{g}\binom{g}{n}\sum_{q=0}^{d-2}2^q\binom{q+n-1}{n-1}+2^{d-1}\sum_{n=1}^{g}\binom{d-1+n-1}{n-1}\binom{g}{n} \\ >\ & 2^{d-1}[g(d-1)+1]+2^{d-1}\sum_{n=1}^{g}\binom{g}{n}\binom{d-1+n-1}{n-1} \qquad \text{[by induction hypothesis]} \\ >\ & 2^{d-1}\left[g(d-1)+1+\binom{g}{1}+\binom{g}{2}d\right] \\ >\ & 2^d(gd+1). \end{aligned}$$

Comparing (6), (7) and (8) we get the inequality in (5). This completes the proof of the theorem. $\square$

4. The Case $g = 2$

In this section we obtain necessary conditions for the bigraded extended Rees algebra to be CM with minimal multiplicity.

Let I_1 and I_2 be ideals of positive height in R. Put $L = I_1 + I_2 + \mathfrak{m}^2$. Recall that if $g = 2$ then

$$\begin{aligned} & e(\mathcal{N}) \\ = \; & \frac{1}{2^d}\left[e(L) + \sum_{q=0}^{d-1}\sum_{j=1}^{2} 2^q e_q(L|I_j) + \sum_{q=0}^{d-1} 2^{d-1-q} \sum_{q_1+q_2=d-1-q} e(L^{[q+1]}|I_1^{[q_1]}|I_2^{[q_2]})\right] \\ \geq \; & \frac{e(L)}{2^d}\left[1 + \sum_{n=1}^{2}\binom{2}{n}\left[\sum_{q=0}^{d-1} 2^q \binom{q+n-1}{n-1} - 2^{d-1}n\right]\right] + e_{d-1}(L|I_1) + e_{d-1}(L|I_2) \\ = \; & (d-1)e(L) + e_{d-1}(L|I_1) + e_{d-1}(L|I_2). \end{aligned} \tag{9}$$

Putting $g = 2$ in Remark 3.3, we get

$$\begin{aligned} & \mu(\mathcal{N}) - \dim\ \mathcal{B}(\mathbf{I}) + 1 \\ = \; & \mu(\mathfrak{m}) + \mu(I_1) + \mu(I_2) - \ell(L/\mathfrak{m}^2) - (d-1) \\ \leq \; & e(\mathfrak{m}) + e_{d-1}(\mathfrak{m}|I_1) + e_{d-1}(\mathfrak{m}|I_2) + 2(d-1) - \ell(L/\mathfrak{m}^2). \end{aligned} \tag{10}$$

Lemma 4.1. *Let I_1 and I_2 be $\mathfrak{m}$-primary ideals in a CM local ring R of positive dimension d. If $\mathcal{B}(\mathbf{I})_{\mathcal{N}}$ is CM with minimal multiplicity, then $\ell(I_1 + I_2 + \mathfrak{m}^2/\mathfrak{m}^2) > 0$.*

Proof. Suppose not. Then $I_1 + I_2 \subseteq \mathfrak{m}^2$. It is easy to see that $e(\mathfrak{m}^n) = n^d e(\mathfrak{m})$ and $e_q(\mathfrak{m}^r|I_j) = r^{d-q}e_q(\mathfrak{m}|I_j)$ for all $q = 1, \ldots, d-1$ for $j = 1, 2$. Hence from (9)

$$\begin{aligned} e(\mathcal{N}) \; &\geq \; (d-1)e(\mathfrak{m}^2) + e_{d-1}(\mathfrak{m}^2|I_1) + e_{d-1}(\mathfrak{m}^2|I_2) \\ &= \; 2^d(d-1)e(\mathfrak{m}) + 2e_{d-1}(\mathfrak{m}|I_1) + 2e_{d-1}(\mathfrak{m}|I_2). \end{aligned} \tag{11}$$

Since $I_1 + I_2 \subseteq \mathfrak{m}^2$, from (10) we get

$$\mu(\mathcal{N}) - \dim\ \mathcal{B}(\mathbf{I}) + 1 \leq e(\mathfrak{m}) + e_{d-1}(\mathfrak{m}|I_1) + e_{d-1}(\mathfrak{m}|I_2) + 2(d-1). \tag{12}$$

Our assumption on $\mathcal{B}(\mathbf{I})_{\mathcal{N}}$ implies that $e(\mathcal{N}) = \mu(\mathcal{N}) - \dim\ \mathcal{B}(\mathbf{I}) + 1$. Hence from (11) and (12) we get

$$[2^d(d-1) - 1]e(\mathfrak{m}) + e_{d-1}(\mathfrak{m}|I_1) + e_{d-1}(\mathfrak{m}|I_2) \leq 2(d-1).$$

Observe that $2^d(d-1) + 1 > 2(d-1)$ for all $d \geq 1$. This leads to a contradiction. □

Lemma 4.2. *Let $(R, \mathfrak{m})$ be a CM local ring of dimension $d \geq 2$. Let I_1 and I_2 be $\mathfrak{m}$-primary ideals in R. If $\mathcal{B}(\mathbf{I})_{\mathcal{N}}$ is CM with minimal multiplicity, then $\ell(I_1+I_2+\mathfrak{m}^2/\mathfrak{m}^2) \leq d$. If $d \geq 3$, then equality holds.*

Proof. Put $L = I_1 + I_2 + \mathfrak{m}^2$. Since $\mathcal{B}(\mathbf{I})_{\mathcal{N}}$ is CM with minimal multiplicity, $e(\mathcal{N}) = \mu(\mathcal{N}) - \dim\ \mathcal{B}(\mathbf{I}) + 1$. From Lemma 3.6, $e_q(L|I_j) \geq e_q(\mathfrak{m}|I_j)$ for $j = 1, 2$. Hence from (9) and (10) we get

$$(d-2)e(\mathfrak{m}) \leq 2(d-1) - \ell(L/\mathfrak{m}^2).$$

Since $e(\mathfrak{m}) \geq 1$, $\ell(L/\mathfrak{m}^2) \leq d$. Let $d \geq 3$. Assume that $\ell(L/\mathfrak{m}^2) \leq d-1$. Then $e(L) \geq e(\mathfrak{m}) + 1$. Once again from (9) and (10) we get

$$\begin{aligned} & 2d - 3 + e(\mathfrak{m}) + e_{d-1}(\mathfrak{m}|I_1) + e_{d-1}(\mathfrak{m}|I_2) \\ \leq\ & (d-1)[e(\mathfrak{m}) + 1] + e_{d-1}(\mathfrak{m}|I_1) + e_{d-1}(\mathfrak{m}|I_2) \\ \leq\ & (d-1)e(L) + e_{d-1}(L|I_1) + e_{d-1}(L|I_2) \\ \leq\ & e(\mathfrak{m}) + 2(d-1) + e_{d-1}(\mathfrak{m}|I_1) + e_{d-1}(\mathfrak{m}|I_2) - \ell(L/\mathfrak{m}^2). \qquad (13) \end{aligned}$$

This gives $\ell(L/\mathfrak{m}^2) \leq 1$. Lemma 4.1 implies that $\ell(L/\mathfrak{m}^2) = 1$. Put $\ell(L/\mathfrak{m}^2) = 1$ in (13). Then equality holds in (13) and hence $e(\mathfrak{m}) = 1$ and $e(L) = 2$. Thus

$$\mu(\mathfrak{m}) = \ell\left(\frac{\mathfrak{m}}{\mathfrak{m}^2}\right) = \ell\left(\frac{R}{L}\right) + \ell\left(\frac{L}{\mathfrak{m}^2}\right) - \ell\left(\frac{R}{\mathfrak{m}}\right) = \ell\left(\frac{R}{L}\right) \leq e(L) = 2.$$

This leads to a contradiction. Hence $\ell(L/\mathfrak{m}^2) = d$. □

Lemma 4.3. *Let $(R, \mathfrak{m})$ be a local ring of positive dimension d. Let I_1 and I_2 be ideals of positive height in R. If $\mathcal{B}(I_1, I_2)_{\mathcal{N}}$ is CM with minimal multiplicity, then $r(I_1) \leq 1$ and $r(I_2) \leq 1$.*

Proof. Let J_i be a minimal reduction of I_i, $(i = 1, 2)$. Then $\mathcal{J} = (t_1^{-1}, t_2^{-1}, \mathfrak{m}, J_1t_1, J_2t_2)$ is a reduction of $\mathcal{N}$. Since $\mathcal{B}(I_1, I_2)_{\mathcal{N}}$ is CM with minimal multiplicity, $J\mathcal{N} = \mathcal{N}^2$ [Sa1, Theorem 1]. Comparing the graded components of $\mathcal{J}\mathcal{N}$ and $\mathcal{N}^2$ we get $J_1I_1 + \mathfrak{m}I_1^2 = I_1^2$ and $J_2I_2 + \mathfrak{m}I_2^2 = I_2^2$. By Nakayama's lemma, $J_1I_1 = I_1^2$ and $J_2I_2 = I_2^2$. □

We are now ready to prove the main results of this section.

Theorem 4.4. *Let $(R, \mathfrak{m})$ be a CM local ring of dimension $d \geq 3$. Put $L = I_1 + I_2 + \mathfrak{m}^2$. Suppose $\mathcal{B}(\mathbf{I})_{\mathcal{N}}$ is CM with minimal multiplicity. Then*

(1) *R is a regular local ring;*

(2) *For* $j = 1, 2$*:*
   (a) $\mu(I_j) = e_{d-1}(\mathfrak{m}|I_j) + d - 1$*;*
   (b) $e_q(L|I_j) = 1$ *for all* $q = 0, \dots d-2$*;*
   (c) $r(I_j) \leq 1$.

Proof. Put $L = I_1 + I_2 + \mathfrak{m}^2$. Recall that

$$\begin{aligned} & e(\mathcal{N}) \\ = \;& \frac{1}{2^d}\left[e(L) + \sum_{q=0}^{d-1} 2^q \left[e_q(L|I_1) + e_q(L|I_2)\right] + \sum_{q=0}^{d-1} 2^{d-1-q} \sum_{q_1+q_2=d-1-q} e(L^{[q+1]}|I_1^{[q_1]}|I_2^{[q_2]})\right] \\ \geq \;& (d-1)e(\mathfrak{m}) + e_{d-1}(\mathfrak{m}|I_1) + e_{d-1}(\mathfrak{m}|I_2) \end{aligned} \tag{14}$$

and

$$\begin{aligned} & \mu(\mathcal{N}) - \dim\ \mathcal{B}(\mathbf{I}) + 1 \\ \leq \;& e(\mathfrak{m}) + e_{d-1}(\mathfrak{m}|I_1) + e_{d-1}(\mathfrak{m}|I_2) + 2(d-1) - \ell(L/\mathfrak{m}^2) \\ = \;& e(\mathfrak{m}) + e_{d-1}(\mathfrak{m}|I_1) + e_{d-1}(\mathfrak{m}|I_2) + d - 2 \qquad \text{[by Lemma 4.2]}. \end{aligned} \tag{15}$$

Since $\mathcal{B}(\mathbf{I})_{\mathcal{N}}$ is CM with minimal multiplicity, $e(\mathcal{N}) = \mu(\mathcal{N}) - \dim\ \mathcal{B}(\mathbf{I}) + 1$. Hence from (14) and (15) we get $(d-2)e(\mathfrak{m}) \leq d - 2$. This implies that $e(\mathfrak{m}) = 1$. Hence equality holds in (14) and (15). As a consequence for $j = 1, 2$ and $q = 0, \dots, d-2$ we have that $\mu(I_j) = e_{d-1}(\mathfrak{m}|I_j) + d - 1$ and $e_q(L|I_j) = 1$. By Lemma 4.3, $r(I_j) \leq 1$ $(j = 1, 2)$. □

Theorem 4.5. *Let* $(R, \mathfrak{m})$ *be a CM local ring of dimension* $d = 2$*. Assume that* $\ell(I_1 + I_2 + \mathfrak{m}^2/\mathfrak{m}^2) = 2$*. Suppose* $\mathcal{B}(\mathbf{I})_{\mathcal{N}}$ *is CM with minimal multiplicity. Then*

(1) R *has minimal multiplicity;*
(2) *For* $j = 1, 2$*:*
   (a) $\mu(I_j) = e_1(\mathfrak{m}|I_j) + 1$*;*
   (b) $r(I_j) \leq 1$.

Proof. The proof of the theorem is similar to the proof of Theorem 4.4. □

5. Special cases and Examples

We recall a result on minimal multiplicity.

Remark 5.1. [Ve2], (2.3) Let $(R, \mathfrak{m})$ be a d-dimensional local ring. If R satisfies the equation of minimal multiplicity, then R is CM if and only if $J\mathfrak{m} = \mathfrak{m}^2$ for some minimal reduction J of $\mathfrak{m}$.

Theorem 5.2. *Let $(R, \mathfrak{m})$ be a CM local ring of positive dimension d. Let r be a positive integer.*

(1) *If $d = 1$, then $\mathcal{B}(\mathfrak{m}, \mathfrak{m}^r)_{\mathcal{N}}$ is CM with minimal multiplicity if and only if R has minimal multiplicity.*
(2) *If $d = 2$, then $\mathcal{B}(\mathfrak{m}, \mathfrak{m}^r)_{\mathcal{N}}$ is CM with minimal multiplicity if and only if R is a regular local ring.*
(3) *If $d \geq 3$, then $\mathcal{B}(\mathfrak{m}, \mathfrak{m}^r)_{\mathcal{N}}$ is CM with minimal multiplicity if and only if R is a regular local ring and $r = 1$.*

Proof. The necessary part can be easily verified for $d = 1$. If $d = 2$, it follows from Theorem 4.5(2a). Let $d \geq 3$. Since $\mathcal{B}(\mathfrak{m}, \mathfrak{m}^r)_{\mathcal{N}}$ is CM with minimal multiplicity, by Theorem 4.4, R is a regular local ring and $\mu(\mathfrak{m}^r) = e_{d-1}(\mathfrak{m}|\mathfrak{m}^r) + d - 1 = r^{d-1} + d - 1$. It is well-known that $\mu(\mathfrak{m}^r) = \binom{r+d-1}{d-1}$. It is easy to verify by induction on d that $\binom{r+d-1}{d-1} > r^{d-1} + d - 1$ for all $d \geq 3$ and for all $r > 1$. Hence $r = 1$.

We now prove the sufficiency. With the assumptions in the theorem it is easy to see that the equation of minimal multiplicity holds for all $d \geq 1$. Let $\mathcal{J} = (t_1^{-1}, x_1^r t_2 + t_2^{-1}, x_d t_1, x_i t_1 + x_{i+1}^r t_2; 1 \leq i \leq d-1)$ (put $r = 1$ for $d \geq 3$). Then $\mathcal{J}\mathcal{N} = \mathcal{N}^2$. In view of Remark 5.1, $\mathcal{B}(\mathfrak{m}, \mathfrak{m}^r)_{\mathcal{N}}$ is CM with minimal multiplicity. $\square$

Theorem 5.3. *Let $(R, \mathfrak{m})$ be a CM local ring of dimension $d \geq 2$. Let I be an $\mathfrak{m}$-primary parameter ideal in R. Then $\mathcal{B}(\mathfrak{m}, I)_{\mathcal{N}}$ is CM with minimal multiplicity if and only if R is a regular local ring and $\ell(I + \mathfrak{m}^2/\mathfrak{m}^2) \geq d - 1$.*

Proof. Suppose $\mathcal{B}(\mathfrak{m}, I)_{\mathcal{N}}$ is CM with minimal multiplicity. Then $e_{d-1}(\mathfrak{m}|I) = \mu(I) - d + 1 = 1$ (Theorem 4.4, Theorem 4.5). This implies that $e(\mathfrak{m}) \leq e_{d-1}(\mathfrak{m}|I) = 1$ and hence $e(\mathfrak{m}) = 1$. By a result of Rees [R3], there exists $x_1, \ldots, x_{d-1} \in I$ and $x_d \in \mathfrak{m}$ such that $e_{d-1}(\mathfrak{m}|I) = e(x_1, \ldots, x_d) = 1$. Hence $\mathfrak{m} = (x_1, \ldots, x_d)$ and $\ell(I + \mathfrak{m}^2/\mathfrak{m}^2) \geq d - 1$.

Conversely, since R is a regular local ring

$$\mu(\mathcal{N}) - \dim\ \mathcal{B}(\mathfrak{m}, I)_{\mathcal{N}} + 1 = 2 + \mu(\mathfrak{m}) + \mu(I) - (d + 2) + 1 = d + 1.$$

Since $\ell(I+\mathfrak{m}^2/\mathfrak{m}^2) \geq d-1$, there exists a regular system of parameters $x_1, \ldots, x_d$ in R such that $I = (x_1, \ldots, x_{d-1}, x_d^r)$. This implies $1 \leq e_q(\mathfrak{m}|I) \leq e(x_1, \ldots, x_d) = 1$ for $q = 0, \ldots d-1$ (Lemma 3.6). Thus

$$e(\mathcal{N}) = \frac{1}{2^d}\left[1 + \sum_{q=0}^{d-1} 2^{q+1} + \sum_{q=0}^{d-1} 2^{d-1-q}(d-q)\right] = d+1.$$

Hence $e(\mathcal{N}) = \mu(\mathcal{N}) - \dim\ \mathcal{B}(\mathfrak{m}, I)_{\mathcal{N}} + 1$. Let

$$\mathcal{J} = (t_1^{-1}, x_1 t_2 + t_2^{-1}, x_d t_1, x_i t_1 + x_{i+1} t_2; 1 \leq i \leq d-2, x_{d-1} t_1 + x_d^r t_2).$$

Then $\mathcal{J}\mathcal{N} = \mathcal{N}^2$. In view of Remark 5.1, $\mathcal{B}(\mathfrak{m}, I)_{\mathcal{N}}$ is CM with minimal multiplicity. □

Remark 5.4. In [Ve1, pg. 3015], J.K. Verma asked the following question: If $(R, \mathfrak{m})$ is a CM local ring, I is any ideal in R, $\mathcal{B}(I)_{\mathcal{N}}$ is CM with minimal multiplicity, then is it true that R has minimal multiplicity. This question does not have an affirmative answer in general. The following example shows that there exist extended Rees algebras which are CM with minimal multiplicity even though R does not have minimal multiplicity. For details on this example the author is requested to see [D1, Example 4.2.8, Example 4.2.9].

Example 5.5. Let $R = k[[x^4, x^5, x^7]]$ where x is an indeterminate, $\mathfrak{m} = (x^4, x^5, x^7)$, $I_1 = (x^4, \mathfrak{m}^2)$, $I_2 = (\mathfrak{m}^2)$. Then R is a CM ring which does not have minimal multiplicity, but $\mathcal{B}(I_1, I_2)_{\mathcal{N}}$ and $\mathcal{B}(I_1)_{\mathcal{N}}$ are CM with minimal multiplicity. □

References

[D1] D'Cruz, C.:*Multigraded Rees algebras of $\mathfrak{m}$-primary ideals in local rings of dimension greater than one*, J. Pure Appl. Algebra **155** (2000) 131-137.

[D2] D'Cruz, C.: *A formula for the multiplicity of the multigraded Rees algebra*, Comm. in Algebra **31** (2003), no. 6, 2573–2585.

[HHRT] Herrmann, M., Hyry, E., Ribbe, J. and Tang, Z.: *Reduction numbers and multiplicities of multigraded structures*, J. Algebra **197** (1997), 311-341.

[Hu] Huneke, C.: *On the associated graded ring of an ideal*, Illinois J. Math. 26 (1982), no. 1, 121–137.

[KV] Katz, D. and Verma, J. K.: *Extended Rees algebras and mixed multiplicities*, Math. Z. **202** (1989) 111-128.

[NR] Northcott, D. G. and Rees, D: *Reductions of ideals in local rings*, Proc. Cambridge Philos. Soc. **50** (1954), 145-158.

[R1] Rees, D.: *$\mathcal{A}$-Transforms of local rings and a theorem on multiplicities of ideals*, Proc. Cambridge Philos. Soc. **57** (1961), 8-17.

[R2] Rees, D.: *Rings associated with ideals and analytic spread*, Math. Proc. Cambridge Philos. Soc. **89** (1981), 423-432.

[R3] Rees, D.: *Generalizations of reductions and mixed multiplicities*, J. London Math. Soc. **29** (1984), 397-414.

[Sa1] Sally, J. D.: *On the associated graded ring of a local Cohen-Macaulay ring*, J. Math. Kyoto Univ. **17** (1977), 19-21.

[Sa2] Sally, J. D.: *"Number of generators of ideals in local rings"*, Lecture Notes in Pure and Applied Mathematics, Vol. 35, Dekker, New York, 1978.

[Sw] Swanson, I.: *Mixed multiplicities, joint reductions and quasi-unmixed local rings*, J. London Math. Soc. **48** (1993), 1-14.

[T] Teissier, B.: *Cycles èvanescents, section planes, et conditions de Whitney, Singularitiés à Cargése*, 1972. Astérisque **7-8** (1973) 285-362.

[Val] Valla, G.: *Certain graded algebras are always Cohen-Macaulay*, J. Algebra **42** (1976) 537-548.

[Ve1] Verma, J. K.: *Rees algebras with minimal multiplicity*, Comm. Algebra **17** (1989), 2999-3024.

[Ve2] Verma, J. K.: *Rees algebra of parameter ideals*, J. Pure Appl. Algebra **61** (1989), 99-106.

[Ve3] Verma, J. K.: *Joint reductions and Rees algebras*, Math. Proc. Cambridge Philos. Soc. **109** (1991), 335-342.

[Ve4] Verma, J. K.: *Rees algebras of contracted ideals in two-dimensional regular local rings*, J. Algebra **141** (1991) 1-10.

[Ve5] Verma, J. K.: *Multigraded Rees algebras and mixed multiplicities*, J. Pure Appl. Algebra **77** (1992), 219-228.

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